

## Isometric Fluctuation Relations for Equilibrium States with Broken Symmetry

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We derive a set of isometric fluctuation relations, which constrain the order parameter fluctuations in finite-size systems at equilibrium and in the presence of a broken symmetry. These relations are exact and should apply generally to many condensed-matter physics systems. Here, we establish these relations for magnetic systems and nematic liquid crystals in a symmetry-breaking external field, and we illustrate them on the Curie-Weiss and the XY models. Our relations also have implications for spontaneous symmetry breaking, which are discussed.

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Away from equilibrium, the second law of thermodynamics quantifies the breaking of the time-reversal symmetry due to energy dissipation, as observed on macroscales. Yet, the microscopic equations of motion are fully reversible and this microreversibility has fundamental implications such as the Onsager reciprocal relations in regimes close to equilibrium, as well as the so-called fluctuation relations, which are also valid further away from equilibrium [1–7].

In this context, Hurtado *et al.* have uncovered a remarkable extension, which they dubbed isometric fluctuation relations, by considering the symmetry under both time reversal and spatial rotations in nonequilibrium fluids [8]. These results hint at the possibility that all the fundamental symmetries continue to manifest themselves in the fluctuations, even if these symmetries are broken by external constraints. This concerns not only systems driven away from equilibrium, but also equilibrium systems described by Gibbsian canonical distributions, as shown by one of us for discrete symmetries such as spin reversal [9]. In this regard, we may wonder whether such fluctuation relations would hold for general broken symmetries in equilibrium systems. The issue is of importance given the central role played by symmetry-breaking phenomena in physics [10,11]. A symmetry may be broken spontaneously if the ground state has a lower symmetry than the Hamiltonian, or explicitly if a perturbation  $H_1$  is added to a Hamiltonian  $H_0$  where  $H_1$  is less symmetric than  $H_0$ . In either case, do the fluctuations of the order parameter leave a footprint of the symmetry that is broken?

The purpose of the present Letter is to answer this fundamental question in the affirmative by proving that, for equilibrium systems, whenever a symmetry is broken by an external field, the probability distribution of the fluctuations obeys an isometric fluctuation relation. Remarkably, this relation is exact already for finite systems. This result

established for magnetic systems and nematic liquid crystals, is illustrated for the Curie-Weiss and XY models of ferromagnetism. We then discuss implications of our result for spontaneous symmetry breaking (SSB).

Let us consider a system composed of  $N$  Heisenberg spins  $\boldsymbol{\sigma} = \{\boldsymbol{\sigma}_i\}_{i=1}^N$ , where the individual spins take discrete or continuous values such that  $\boldsymbol{\sigma}_i \in \mathbb{R}^d$  and  $\|\boldsymbol{\sigma}_i\| = 1$ . The order parameter of this system is the magnetization  $\mathbf{M}_N(\boldsymbol{\sigma}) = \sum_{i=1}^N \boldsymbol{\sigma}_i$ , and the Hamiltonian of the system is assumed to be of the form  $H_N(\boldsymbol{\sigma}; \mathbf{B}) = H_N(\boldsymbol{\sigma}; \mathbf{0}) - \mathbf{B} \cdot \mathbf{M}_N(\boldsymbol{\sigma})$ , where  $\mathbf{B}$  is the magnetic field. Let us also introduce the probability  $P_{\mathbf{B}}(\mathbf{M})$  that the magnetization takes the value  $\mathbf{M} = \mathbf{M}_N(\boldsymbol{\sigma})$  as  $P_{\mathbf{B}}(\mathbf{M}) = \langle \delta[\mathbf{M} - \mathbf{M}_N(\boldsymbol{\sigma})] \rangle_{\mathbf{B}}$ , where  $\delta(\cdot)$  denotes the Dirac delta distribution and  $\langle \cdot \rangle_{\mathbf{B}}$  the statistical average over Gibbs' canonical measure at the inverse temperature  $\beta$  [12]. First, we establish a general identity between the distribution of the order parameter in the field,  $P_{\mathbf{B}}(\mathbf{M})$ , and the same distribution in the absence of the field,  $P_0(\mathbf{M})$ :

$$\begin{aligned} P_{\mathbf{B}}(\mathbf{M}) &= \frac{1}{Z_N(\mathbf{B})} \sum_{\boldsymbol{\sigma}} e^{-\beta H_N(\boldsymbol{\sigma}; \mathbf{0}) + \beta \mathbf{B} \cdot \mathbf{M}_N(\boldsymbol{\sigma})} \delta[\mathbf{M} - \mathbf{M}_N(\boldsymbol{\sigma})], \\ &= \frac{1}{Z_N(\mathbf{B})} e^{\beta \mathbf{B} \cdot \mathbf{M}} \sum_{\boldsymbol{\sigma}} e^{-\beta H_N(\boldsymbol{\sigma}; \mathbf{0})} \delta[\mathbf{M} - \mathbf{M}_N(\boldsymbol{\sigma})], \\ &= \frac{Z_N(\mathbf{0})}{Z_N(\mathbf{B})} e^{\beta \mathbf{B} \cdot \mathbf{M}} P_0(\mathbf{M}), \end{aligned} \quad (1)$$

where  $Z_N(\mathbf{B})$  is the partition function. We notice that this identity holds even if the Hamiltonian  $H_N(\boldsymbol{\sigma}; \mathbf{0})$  has no particular symmetry.

Now, we suppose that, in the absence of the field, the Hamiltonian  $H_N(\boldsymbol{\sigma}; \mathbf{0})$  is invariant under a symmetry group  $G$ , which can be discrete or continuous. This means that  $H_N(\boldsymbol{\sigma}^g; \mathbf{0}) = H_N(\boldsymbol{\sigma}; \mathbf{0})$ , where  $\boldsymbol{\sigma}^g = \{\mathbf{R}_g \cdot \boldsymbol{\sigma}_i\}_{i=1}^N$ , and  $\mathbf{R}_g$  is a representation of the member  $g$  of the group  $G$  such that

$|\det \mathbf{R}_g| = 1$ . The consequence is that the probability distribution of the magnetization has this symmetry in the absence of the magnetic field since summing over the microstates  $\boldsymbol{\sigma}$  or their symmetry transforms  $\boldsymbol{\sigma}^g$  are equivalent for every  $g \in G$  so that

$$\begin{aligned} P_{\mathbf{0}}(\mathbf{M}) &= \frac{1}{Z_N(\mathbf{0})} \sum_{\boldsymbol{\sigma}^g} e^{-\beta H_N(\boldsymbol{\sigma}^g; \mathbf{0})} \delta[\mathbf{M} - \mathbf{M}_N(\boldsymbol{\sigma}^g)], \\ &= \frac{1}{Z_N(\mathbf{0})} \sum_{\boldsymbol{\sigma}} e^{-\beta H_N(\boldsymbol{\sigma}; \mathbf{0})} \delta[\mathbf{M} - \mathbf{R}_g \cdot \mathbf{M}_N(\boldsymbol{\sigma})], \\ &= P_{\mathbf{0}}(\mathbf{R}_g^{-1} \cdot \mathbf{M}). \end{aligned} \quad (2)$$

Combining Eqs. (1) and (2), one obtains the fluctuation relation

$$P_{\mathbf{B}}(\mathbf{M}) = P_{\mathbf{B}}(\mathbf{M}') e^{\beta \mathbf{B} \cdot (\mathbf{M} - \mathbf{M}')}, \quad (3)$$

with  $\mathbf{M}' = \mathbf{R}_g^{-1} \cdot \mathbf{M}$  for all  $g \in G$ . When  $\mathbf{R}_g$  represents a rotation,  $\|\mathbf{M}\| = \|\mathbf{M}'\|$ , hence the name isometric fluctuation relation. This relation includes as a particular case the fluctuation relation derived in Ref. [9] when  $\mathbf{M}' = -\mathbf{M}$ . In analogy with the nonequilibrium case, a corollary of this relation can be obtained by introducing the Kullback-Leibler (KL) divergence of the distributions  $P_{\mathbf{B}}(\mathbf{M})$  and  $P_{\mathbf{B}}(\mathbf{M}')$ . The positivity of the KL divergence leads to the second-law-like inequality

$$\mathbf{B} \cdot \langle \mathbf{M} \rangle_{\mathbf{B}} \geq \mathbf{B} \cdot \langle \mathbf{M}' \rangle_{\mathbf{B}}, \quad (4)$$

where  $\langle \cdot \rangle_{\mathbf{B}}$  represents an average with respect to the distribution  $P_{\mathbf{B}}(\mathbf{M})$ .

In Eq. (3), the distribution of the order parameter is compared to the distribution of the rotated order parameter in the same magnetic field. Another possibility is to fix the order parameter and rotate the magnetic field. A similar derivation leads to

$$P_{\mathbf{B}}(\mathbf{M}) = P_{\mathbf{B}'}(\mathbf{M}) e^{\beta (\mathbf{B} - \mathbf{B}') \cdot \mathbf{M}}, \quad (5)$$

where  $\mathbf{B}' = \mathbf{R}_g^T \cdot \mathbf{B}$  for all  $g \in G$ , with  $T$  denoting the transpose. We emphasize that the fluctuation relations (3) and (5) hold exactly in finite systems.

The isometric fluctuation relation (3) also holds locally for a spatially varying magnetization density  $\mathbf{m}(\mathbf{r})$  and magnetic field  $\mathbf{B}(\mathbf{r})$ . To show this, it is needed to proceed by coarse graining the magnetization density  $\mathbf{m}(\mathbf{r}) = \sum_{i=1}^N \boldsymbol{\sigma}_i \delta(\mathbf{r} - \mathbf{r}_i)$ , where  $\mathbf{r}_i$  is the location of spin  $\boldsymbol{\sigma}_i$ . By adapting the derivation of Eq. (3), one then finds

$$P_{\mathbf{B}}[\mathbf{m}(\mathbf{r})] = P_{\mathbf{B}'}[\mathbf{m}'(\mathbf{r})] e^{\beta \int d\mathbf{r} \mathbf{B}(\mathbf{r}) \cdot [\mathbf{m}(\mathbf{r}) - \mathbf{m}'(\mathbf{r})]}, \quad (6)$$

where  $P_{\mathbf{B}}[\mathbf{m}(\mathbf{r})]$  is the probability functional of the magnetization density  $\mathbf{m}(\mathbf{r})$  and  $\mathbf{m}'(\mathbf{r}) = \mathbf{R}_g^{-1} \cdot \mathbf{m}(\mathbf{r})$  (see the Supplemental Material [13] for details).

In the infinite-system limit  $N \rightarrow \infty$ , these fluctuation relations have their counterparts in terms of large-deviation functions [17–19]. By defining the magnetization per spin  $\mathbf{m} = \mathbf{M}/N$ , one can introduce a large-deviation function  $\Phi_{\mathbf{B}}(\mathbf{m})$  such that

$$P_{\mathbf{B}}(\mathbf{M}) = A_N(\mathbf{m}) e^{-N \Phi_{\mathbf{B}}(\mathbf{m})}, \quad (7)$$

where  $A_N(\mathbf{m})$  is a prefactor which has a negligible contribution to  $\Phi_{\mathbf{B}}(\mathbf{m})$  in the limit  $N \rightarrow \infty$ . As a result, Eq. (3) implies the following symmetry relation for the large-deviation function:

$$\Phi_{\mathbf{B}}(\mathbf{m}) - \Phi_{\mathbf{B}}(\mathbf{m}') = \beta \mathbf{B} \cdot (\mathbf{m}' - \mathbf{m}). \quad (8)$$

It is important to appreciate that the function  $\Phi_{\mathbf{B}}(\mathbf{m})$  characterizes the equilibrium fluctuations of the order parameter which are in general non-Gaussian. This function can be expressed in terms of the Helmholtz free energy per spin,  $f(\mathbf{B}) = -\beta^{-1} \ln Z_N(\mathbf{B})/N$ , and of its Legendre transform  $\Phi_{\mathbf{0}}(\mathbf{m})$  using Eqs. (1) and (7), as  $\Phi_{\mathbf{B}}(\mathbf{m}) = \Phi_{\mathbf{0}}(\mathbf{m}) - \beta \mathbf{B} \cdot \mathbf{m} - \beta f(\mathbf{B}) + \beta f(\mathbf{0})$  [9]. Unlike the Helmholtz free energy  $f(\mathbf{B})$  or its Legendre transform  $\Phi_{\mathbf{0}}(\mathbf{m})$ , the function  $\Phi_{\mathbf{B}}(\mathbf{m})$  depends on both thermodynamically conjugated variables  $\mathbf{m}$  and  $\mathbf{B}$ .

We may also introduce the cumulant generating function for the magnetization:

$$\Gamma_{\mathbf{B}}(\boldsymbol{\lambda}) \equiv \lim_{N \rightarrow \infty} -\frac{1}{N} \ln \langle e^{-\boldsymbol{\lambda} \cdot \mathbf{M}_N(\boldsymbol{\sigma})} \rangle_{\mathbf{B}}, \quad (9)$$

which is the Legendre-Fenchel transform of the function  $\Phi_{\mathbf{B}}(\mathbf{m})$  defined by Eq. (7). As a consequence of the isometric fluctuation relation (3), the generating function (9) obeys the symmetry relation  $\Gamma_{\mathbf{B}}(\boldsymbol{\lambda}) = \Gamma_{\mathbf{B}}[\beta \mathbf{B} + \mathbf{R}_g^T \cdot (\boldsymbol{\lambda} - \beta \mathbf{B})]$  for all  $g \in G$ . In the particular case of the inversion  $\mathbf{R}_g = -\mathbf{1}$ , we find that

$$\Gamma_{\mathbf{B}}(\boldsymbol{\lambda}) = \Gamma_{\mathbf{B}}(2\beta \mathbf{B} - \boldsymbol{\lambda}). \quad (10)$$

The first cumulant, which is the average magnetization per spin, is thus given by

$$\langle \mathbf{m} \rangle_{\mathbf{B}} = \frac{\partial \Gamma_{\mathbf{B}}}{\partial \boldsymbol{\lambda}}(\mathbf{0}) = -\frac{\partial \Gamma_{\mathbf{B}}}{\partial \boldsymbol{\lambda}}(2\beta \mathbf{B}), \quad (11)$$

which has fundamental implications about SSB. Indeed, as long as the cumulant generating function (9) remains analytic in the variables  $\boldsymbol{\lambda}$  (which is necessarily the case in a finite system), the average magnetization has to vanish in the absence of external field because  $\langle \mathbf{m} \rangle_{\mathbf{0}} = \partial_{\boldsymbol{\lambda}} \Gamma_{\mathbf{0}}(\mathbf{0}) = -\partial_{\boldsymbol{\lambda}} \Gamma_{\mathbf{0}}(\mathbf{0}) = -\langle \mathbf{m} \rangle_{\mathbf{0}} = 0$ , as implied by Eq. (11). Because of the thermodynamic limit  $N \rightarrow \infty$ , the generating function may not be analytic, allowing a spontaneous magnetization  $\langle \mathbf{m} \rangle_{\mathbf{0}} \neq 0$  in the absence of external field, and thus the possibility of SSB. In this case, in view of the symmetry (10), the simplest possible form of the generating function near  $\boldsymbol{\lambda} = \beta \mathbf{B}$  is

$$|\Gamma_{\mathbf{B}}(\boldsymbol{\lambda}) - \Gamma_{\mathbf{B}}(\beta\mathbf{B})|_{T_c} \sim \|\boldsymbol{\lambda} - \beta\mathbf{B}\|^{1+1/\delta} \quad (12)$$

at the critical temperature  $T_c$ , in order for the critical magnetization to scale as  $\|\langle \mathbf{m} \rangle_{\mathbf{B}}\|_{T_c} \sim \|\mathbf{B}\|^{1/\delta}$  with the critical exponent  $\delta = 3$  in the mean-field models, or  $\delta = 15$  in the two-dimensional Ising model [20–22]. The universal scaling behavior (12) establishes the nonanalyticity of the generating function, which allows for the possibility of a nonvanishing spontaneous magnetization in the thermodynamic limit.

Now, we study a selection of illustrative examples of magnetic systems. Let us start by considering  $N$  Heisenberg spins with Curie-Weiss interaction. The Hamiltonian of this system is

$$H_N(\boldsymbol{\sigma}; \mathbf{B}) = -\frac{J}{2N} \mathbf{M}_N(\boldsymbol{\sigma})^2 - \mathbf{B} \cdot \mathbf{M}_N(\boldsymbol{\sigma}). \quad (13)$$

The distribution of the order parameter is

$$P_{\mathbf{B}}(\mathbf{M}) = \frac{1}{Z_N(\mathbf{B})} e^{(\beta J/2N)\mathbf{M}^2 + \beta\mathbf{B}\cdot\mathbf{M}} C_N(\mathbf{M}), \quad (14)$$

where the function  $C_N(\mathbf{M}) = \sum_{\boldsymbol{\sigma}} \delta[\mathbf{M} - \mathbf{M}_N(\boldsymbol{\sigma})]$  represents the number of microstates with a given magnetization  $\mathbf{M}$ . This number, which has to be rotationally invariant, is related by  $C_N = e^{S_N/k}$  to the entropy function  $S_N(\mathbf{M})$  and Boltzmann constant  $k$ . Using large-deviation theory [13,17,19], one explicitly obtains this entropy in the form of  $S_N(N\mathbf{m}) = -kNI(m)$  with  $m = \|\mathbf{m}\|$  and

$$I(m) = m\mathcal{L}^{-1}(m) - \ln \frac{\sinh[\mathcal{L}^{-1}(m)]}{\mathcal{L}^{-1}(m)}, \quad (15)$$

where  $\mathcal{L}^{-1}$  is the inverse of the Langevin function  $\mathcal{L}(x) = \coth(x) - 1/x$ , a result which also follows from a standard mean-field approach [23]. Combining Eqs. (14) and (15), the large-deviation function defined in Eq. (7) is  $\Phi_{\mathbf{B}}(\mathbf{m}) = I(m) - \beta Jm^2/2 - \beta\mathbf{B} \cdot \mathbf{m}$ . The prefactor  $A_N(\mathbf{m})$  of Eq. (7) is calculated in the Supplemental Material [13]. For this model, it is straightforward to check that this large-deviation function satisfies the symmetry relation of Eq. (8). In Fig. 1, we show the distribution  $P_{\mathbf{B}}(N\mathbf{m})$  as a function of the components  $(m_x, m_y)$  of the magnetization per spin. In the absence of external field, the probability distribution is spherically symmetric, in which case spontaneous symmetry breaking occurs below the critical temperature. Figure 2 depicts the cumulant generating function (9) below and above the critical temperature. If this function is analytic in the paramagnetic phase above the critical temperature, it is no longer the case below the critical temperature in the ferromagnetic phase where the function presents a discontinuity at the symmetry point  $\boldsymbol{\lambda} = \beta\mathbf{B}$  in its derivatives with respect to the parameters  $\boldsymbol{\lambda}$ . As aforementioned, this nonanalyticity is at the origin of the spontaneous magnetization in the ferromagnetic phase.

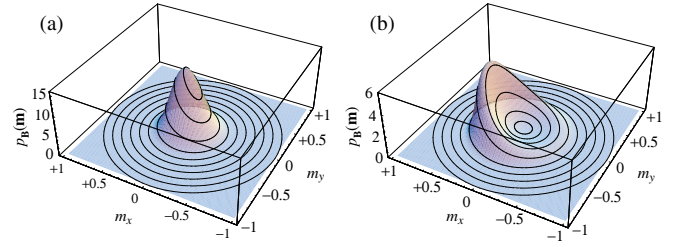


FIG. 1 (color online). Probability density  $p_{\mathbf{B}}(\mathbf{m}) = N^3 P_{\mathbf{B}}(N\mathbf{m})$  of the magnetization per spin  $\mathbf{m} = \mathbf{M}/N = (m_x, m_y, m_z = 0)$  for the three-dimensional Curie-Weiss model in the magnetic field  $\mathbf{B} = (B, 0, 0)$  with  $B = 0.01$ ,  $J = 1$ , and  $N = 100$  at the rescaled inverse temperatures (a)  $\beta J = 2.7$  in the paramagnetic phase and (b)  $\beta J = 3.3$  in the ferromagnetic phase. The lines depict the contours of  $\|\mathbf{m}\| = 0.1, 0.2, \dots, 1.0$  where the isometric fluctuation relation (3) holds [13].

At the critical temperature, the generating function has the universal scaling behavior (12) with  $\delta = 3$ , as it should for this mean-field model [13].

We proceed by investigating the more complex  $XY$  model, in light of our findings on isometric fluctuation relations. In this much-studied model, topological defects unbind above the Kosterlitz-Thouless transition temperature  $T_{KT}$ , where the order changes from quasi-long range to short range [24]. The Hamiltonian of the  $XY$  model in an external magnetic field  $\mathbf{B} = (B_x, B_y)$  is given by

$$H_N(\boldsymbol{\theta}; \mathbf{B}) = -J \sum_{\langle i,j \rangle} \cos(\theta_i - \theta_j) - \mathbf{B} \cdot \mathbf{M}, \quad (16)$$

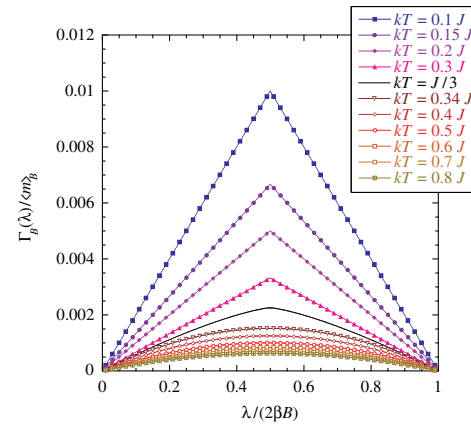


FIG. 2 (color online). Cumulant generating function (9) of the magnetization in the three-dimensional Curie-Weiss model for the magnetic field  $\mathbf{B} = (B, 0, 0)$  with  $B = 0.001$ ,  $J = 1$ , and different temperatures across criticality. The generating function is rescaled by the average magnetization  $\langle m \rangle_B$  in the direction of the external field and plotted versus the rescaled parameter  $\lambda/(2\beta B)$  in the same direction  $\boldsymbol{\lambda} = (\lambda, 0, 0)$ . The generating function is computed by taking the Legendre-Fenchel transform of the large-deviation function  $\Phi_{\mathbf{B}}(\mathbf{m})$  introduced in Eq. (7) [13]. The isometric fluctuation relation (3) implies the symmetry  $\lambda \rightarrow 2\beta B - \lambda$  of the generating function according to Eq. (10).

on a square lattice with  $L \times L$  sites ( $i, j = 1, 2, \dots, L$ ) with the magnetization  $\mathbf{M} = \sum_i (\cos \theta_i, \sin \theta_i)$ . In the absence of an external field, the Hamiltonian is symmetric under the orthogonal group  $O(2)$ . In view of Eq. (1), the probability distribution of the magnetization in the field can be obtained from the same distribution in the absence of the field  $P_0(\mathbf{M})$ . This quantity is itself related to the probability distribution of the modulus of the magnetization  $Q(M)$  by  $P_0(\mathbf{M}) = Q(M)/(2\pi M)$ . A rich physics is contained in the distribution  $Q(M)$  [25]. In particular, a calculation of this quantity below  $T_{KT}$  has been shown to be numerically very close to a Gumbel distribution [26].

In order to test the isometric fluctuation relations, we have carried out Monte Carlo simulations as shown in Fig. 3. Figure 3(a) represents the quantity  $Q_{\mathbf{B}}(M, \theta) \equiv 2\pi M P_{\mathbf{B}}(\mathbf{M})$  versus the magnetization per spin  $m = M/N$  for different values of the angle  $\theta$ , while Fig. 3(b) depicts the probability distribution compensated by Boltzmann weights:  $f_{\mathbf{B}}(M, \theta) \equiv Q_{\mathbf{B}}(M, \theta)e^{-\beta BM \cos \theta}$ . The coincidence of all the curves is evidence of the isometric fluctuation relation. In addition, we show in the inset a test of the relation using an equivalent form put forward by Hurtado *et al.* [8]:

$$\frac{1}{\beta BM} \ln \frac{Q_{\mathbf{B}}(M, \theta)}{Q_{\mathbf{B}}(M, \theta')} = \cos \theta - \cos \theta'. \quad (17)$$

We have checked that the relation holds at temperatures below  $T_{KT}$ , as well as above.

Besides the Curie-Weiss and XY models, the isometric fluctuation relation applies as well to other magnetic

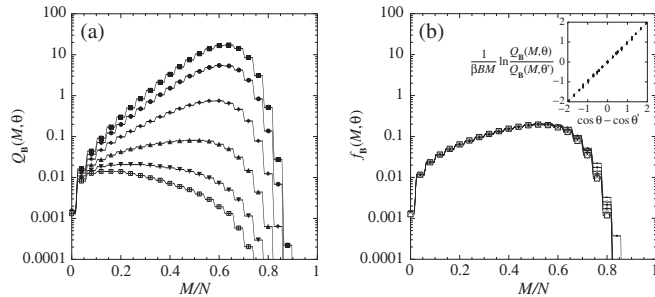


FIG. 3. XY model of two-dimensional magnetism on a square lattice with  $L = 10$  and  $J = 1$  in the external magnetic field  $\mathbf{B} = (0.1, 0)$  at the rescaled inverse temperature  $\beta J = 0.8$  above the critical temperature  $T_{KT}$ . (a) The distribution  $Q_{\mathbf{B}}(M, \theta) \equiv 2\pi M P_{\mathbf{B}}(\mathbf{M})$  versus the modulus of the magnetization per spin  $M/N = \|\mathbf{M}\|/N$  for different values of the angle  $\theta$  separated by  $\Delta\theta = \pi/6$ , with the top curve corresponding to the direction along the magnetic field. (b) The distribution compensated by Boltzmann weights  $f_{\mathbf{B}}(M, \theta) \equiv Q_{\mathbf{B}}(M, \theta) \exp(-\beta BM \cos \theta)$  confirming the isometric fluctuation relation. Inset: Check of the equality between the left- and right-hand sides of Eq. (17). After a transitory run of  $10^4$  spin flips, the statistics is carried out over  $10^8$  values of the magnetization, each one separated by  $10^3$  spin flips.

systems. In particular, the relation can be established using the transfer-matrix method in the case of a 1D classical chain of Heisenberg spins [27].

Beyond magnetic systems, broken symmetry phases are ubiquitous in soft matter systems, such as liquid crystals. These systems are of great interest to study deformations and orientation due to heterogeneities or to the application of external fields. Below, we focus on nematic liquid crystals which can be described by a tensorial order parameter  $\mathbf{Q}$ , or equivalently by a scalar order parameter and a director  $\mathbf{n}$  for uniaxial nematics [28]. Here, we discuss the fluctuations of the tensorial order parameter in a finite ensemble of nematogens.

Let us consider the following general Hamiltonian

$$H_N(\boldsymbol{\sigma}; \mathbf{B}) = H_N(\boldsymbol{\sigma}; \mathbf{0}) - \mathbf{B}^T \cdot \mathbf{Q}_N(\boldsymbol{\sigma}) \cdot \mathbf{B}, \quad (18)$$

with the following traceless tensorial order parameter:

$$\mathbf{Q}_N(\boldsymbol{\sigma}) = \sum_{i=1}^N \left( \boldsymbol{\sigma}_i \otimes \boldsymbol{\sigma}_i^T - \frac{1}{d} \mathbf{1} \right), \quad (19)$$

where now  $\boldsymbol{\sigma}_i \in \mathbb{R}^d$  is a unit vector directed along the axis of the nematogens molecules. The distribution of this tensor is  $P_{\mathbf{B}}(\mathbf{Q}) \equiv \langle \delta[\mathbf{Q} - \mathbf{Q}_N(\boldsymbol{\sigma})] \rangle_{\mathbf{B}}$ , where  $\langle \cdot \rangle_{\mathbf{B}}$  denotes the statistical average over Gibbs' canonical measure. Using a similar derivation as before for a vectorial order parameter, one obtains the following isometric fluctuation relations for the distribution of the tensorial order parameter  $\mathbf{Q}$ :

$$P_{\mathbf{B}}(\mathbf{Q}) = P_{\mathbf{B}}(\mathbf{Q}') e^{\beta \mathbf{B}^T \cdot (\mathbf{Q} - \mathbf{Q}') \cdot \mathbf{B}}, \quad (20)$$

with  $\mathbf{Q}' = \mathbf{R}_g^{-1} \cdot \mathbf{Q} \cdot \mathbf{R}_g^{-1 T}$  for all  $g \in G$ . We will report elsewhere the application of this relation to a variant of the Maier-Saupe model [29] and its extension to the continuum description of long-wavelength fluctuations of the director field  $\mathbf{n}(\mathbf{r})$  [27].

In this Letter, we have obtained isometric fluctuation relations for equilibrium systems. These relations are exact and hold for finite as well as infinite systems. We have shown that the fundamental symmetries of systems undergoing explicit or spontaneous symmetry breaking continue remarkably to manifest themselves in the fluctuations of the order parameter. The fluctuation relations take slightly different forms depending on the particular interaction energy of the system with the symmetry-breaking field, as shown in the examples with magnetic or nematic systems.

We have also shown in Eq. (6) that the symmetry relation holds not only for the global order parameter in a finite system but also locally in spatially extended systems. A potential application of this result for experiments could consist in looking for an asymmetry in the local fluctuations of an order parameter, and in extracting from this asymmetry information about the symmetry-breaking field

$\mathbf{B}(\mathbf{r})$ . This would be the equilibrium analog of recent experiments, in which the asymmetry in nonequilibrium fluctuations have been exploited to estimate the entropy production [30,31]. Another interesting application of this framework, which we plan to pursue, concerns the analysis of fluctuations in the critical regime, which are accessible experimentally [32]. In conclusion, the isometric fluctuation relations point towards a deep connection between fluctuations and symmetries, beyond the distinction between equilibrium and nonequilibrium. This deep link not only sheds a new light on the classic topic of symmetry breaking, but is also likely to be a useful tool for extracting relevant information from fluctuations.

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