Capillary-gravity waves generated by a sudden object motion

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We study theoretically the capillary-gravity waves created at the water-air interface by a small object during a sudden accelerated or decelerated rectilinear motion. We analyze the wave resistance corresponding to the transient wave pattern and show that it is nonzero even if the involved velocity (the final one in the accelerated case, the initial one in the decelerated case) is smaller than the minimum phase velocity \( c_{\text{min}} = 23 \, \text{cm s}^{-1} \). These results might be important for a better understanding of the propulsion of water-walking insects where accelerated and decelerated motions frequently occur. © 2010 American Institute of Physics. [doi:10.1063/1.3430004]

I. INTRODUCTION

If a body (such as a boat or an insect) or an external pressure source moves at the free liquid-air interface, it generates capillary-gravity waves. These are driven by a balance between the liquid inertia and its tendency, under the action of gravity and under surface tension forces, to return to a state of stable equilibrium. For an inviscid liquid of infinite depth, the dispersion relation of capillary-gravity waves, relating the angular frequency \( \omega \) to the wavenumber \( k \), is given by \( \omega^2 = gk + \gamma k^3 / \rho \), where \( \gamma \) is the liquid-air surface tension, \( \rho \) is the liquid density, and \( g \) is the acceleration due to gravity. The energy carried away by the waves is felt by the body (or the pressure source) as a drag \( R_w \), called the wave resistance. In the case of boats or ships, the wave resistance has been well studied in order to design hulls minimizing it. The case of objects that are small compared to the capillary length has been considered only recently.

In a disturbance moving at constant velocity \( V = |V| \) on a rectilinear trajectory, the wave resistance \( R_w \) is zero for \( V < c_{\text{min}} \), where \( c_{\text{min}} = (4g \rho / \gamma)^{1/4} \) is the minimum of the wave velocity \( c(k) = \omega(k)/k = \sqrt{g/k + \gamma k / \rho} \) for capillarity gravity waves. Effectively, in the frame moving with that object, the problem must be stationary. That is true only if the radiated waves have a phase velocity \( c(k) \) equal to the object’s velocity \( V \). If \( V < c_{\text{min}} \), no solutions exist, hence no waves, and the wave resistance is zero. For water with \( \gamma = 73 \, \text{mN m}^{-1} \) and \( \rho = 10^3 \, \text{kg m}^{-3} \), one has \( c_{\text{min}} = 0.23 \, \text{m s}^{-1} \) (at room temperature). It was recently shown by Chepelianskii et al. that no such velocity threshold exists for a steady circular motion, for which, even for small velocities, a finite wave drag is experienced by the object. Here we consider the case of a sudden accelerated or decelerated rectilinear motion and show that the transient wave pattern leads to a nonzero wave resistance even if the involved velocity (the final one for the accelerated case, the initial one for the decelerated case) is smaller than the minimum phase velocity \( c_{\text{min}} = 23 \, \text{cm s}^{-1} \). In order to avoid the problem of the relocation of the contact line, we have replaced the partially submerged object by a external pressure distribution. The physical origin of the results presented here is similar to the Cherenkov radiation emitted by accelerated (or decelerated) charged particles and has also been recently discussed in the context of the response of a floating ice sheet to an accelerating line load and in the context of water waves generated by a moving bottom.

II. EQUATIONS OF MOTION

We consider an inviscid, deep liquid with an infinitely extending free surface. To locate a point on the free surface, we introduce a vector \( \mathbf{r} = (x, y) \) in the horizontal plane associated with the equilibrium state of a flat surface. The motion of the disturbance in this plane induces a vertical displacement \( \xi(r, t) \) (Monge representation) of the free surface from its equilibrium position.

Assuming that the liquid equations of motion can be linearized (in the limit of small wave amplitudes), one has

\[
\frac{\partial^2 \hat{\xi}(k, t)}{\partial t^2} + \omega(k)^2 \hat{\xi}(k, t) = -\frac{k \hat{P}_{\text{ext}}(k, t)}{\rho},
\]

where \( \hat{P}_{\text{ext}}(k, t) \) and \( \hat{\xi}(k, t) \) are the Fourier transforms of the pressure distribution and the displacement, respectively. In what follows, we will assume that the pressure distribution is axisymmetric around the point \( \mathbf{r}_0(t) \) (corresponding to the disturbance trajectory). \( \hat{P}_{\text{ext}}(k, t) \) can then be written as \( \hat{P}_{\text{ext}}(k)e^{-ik\mathbf{r}_0(t)} \).

A. Uniform straight motion

Let us first recall the results previously obtained in the case of a uniform straight motion. Such a motion corresponds to \( \mathbf{r}_0(t) = (-Vt, 0) \), where \( V \) is the constant velocity of the disturbance.

Equation (1) then becomes
The above equation corresponds to the equation of a harmonic oscillator forced at angular frequency $V k_c$. We can solve it by looking for solutions with a time-dependence of the form $e^{i V k_c t}$, leading to

$$\dot{\zeta}(k,t) = -\frac{k \hat{P}_{ext}(k)}{\rho \omega(k)^2 - (V k_c)^2} e^{i V k_c t}. $$

Following Havelock, the wave resistance $R_w$ experienced by the moving disturbance is given by

$$R_w = -i \int \frac{dk_1 dk_2}{2\pi^2} \frac{2\pi}{2\pi} \hat{P}_{ext}(k, t) \hat{P}_x(k, t).$$

This expression for the wave resistance represents the total force exerted by the external pressure on the free surface $R_w = \int dxdy P_{ext}(r, t) \nabla \zeta(r, t)$ written in Fourier space. Using Eq. (3) and integrating over the angular variable, one obtains

$$R_w = \int_0^{2\pi} \frac{dk}{2\pi} \frac{2\pi}{2\pi} \frac{k \hat{P}_{ext}(k)}{\rho \omega(k)^2 - (V k_c)^2} e^{-i V k_c t} \mathbf{u}_x,$$

where $\mathbf{u}_x$ is the unit vector along the x-axis. The behavior of $R_w$ as a function of the disturbance velocity $V$ is illustrated in Fig. 1. There we have assumed a pressure disturbance of Lorentzian form with a Fourier transform $\hat{P}_{ext}(k) = P_0 e^{-b k}$, where $b$ is the object size (set to $b = 0.1 k_c^{-1}$). This choice will be taken for the figures throughout this work. The wave resistance is equal to zero for $V < c_{min}$ and presents a discontinuous behavior at $V = c_{min}$ (see Refs. 8 and 13 for a more complete discussion).

**B. Accelerated straight motion**

We now turn to the case where the disturbance—initially at rest—is suddenly set to a uniform motion (characterized by a constant velocity $V$) at time $t = 0$. The corresponding trajectory is given by $r(t) = -V t \theta(-t) \mathbf{u}_x$. As long as the perturbation does not move (i.e., for $t < 0$), the wave resistance is equal to zero. In order to calculate the wave resistance for $t > 0$, we solve Eq. (2) along with the initial conditions $\zeta(k, t=0) = 0$ and $\partial \zeta(k, t=0)/\partial t = 0$, yielding

$$\dot{\zeta}(k,t) = -\int_0^t d\tau \frac{k \hat{P}_{ext}(k)}{\rho \omega(k)^2 - (V k_c)^2} e^{-i V k_c \tau} \omega(k)(\tau-t).$$

Equation (4) then leads to the following expression for the wave resistance:

$$R_w(t) = \int_0^\infty \frac{dk}{2\pi} \int_0^t \frac{d\tau}{\rho \omega(k)} \frac{k \hat{P}_{ext}(k)}{\rho \omega(k)^2 - (V k_c)^2} e^{-i V k_c \tau} \omega(k) J_1(k V u) \mathbf{u}_x,$$

where $J_1(x)$ is the first Bessel functions of the first kind. In the long time limit, one has

$$\lim_{t \to \infty} \int_0^t d\tau \sin[\omega(k) u] J_1(k V u) \mathbf{u}_x = \frac{\omega(k) \theta[V - c(k)]}{V^2 k_c^2 [1 - c(k)/V]^2},$$

and therefore the wave resistance, Eq. (7), converges to the uniform straight motion result, Eq. (5). The behavior of $R_w(t)$ is represented in Fig. 2 for different values of the disturbance velocity and will be discussed in detail in Sec. III.

**C. Decelerated straight motion**

Let us now consider the case of a disturbance moving with a constant velocity $V$ for $t < 0$ and suddenly set and maintained at rest for $t > 0$. This corresponds to the following trajectory: $r(t) = -V t \theta(-t) \mathbf{u}_x$. As long as $t < 0$, the wave resistance $R_w$ is given by the uniform straight motion result, Eq. (5). In order to calculate the wave resistance for $t > 0$, we solve Eq. (2) along with the initial conditions.
values of the disturbance velocity and will be discussed in detail in section the next section (Sec. III). The negative values of $R_u(t)$ that one may observe in the decelerated motion correspond to a drag force acting on the disturbance along the negative $x$-axis [the disturbance, maintained at rest, is continually interacting with the gravity-dominated waves emitted at $t < 0$ (see below); hence the changes in the sign of $R_u(t)$ in Fig. 3].

III. RESULTS AND DISCUSSION

A. Accelerated straight motion

Figures 2 and 3 represent the behavior of the wave resistance for the accelerated and the decelerated cases, respectively. In order to get a better understanding of the behavior of $R_u(t) = R_u \cdot u_x$, we will perform analytic expansions of Eq. (7) and Eq. (12), respectively. Let us start with the accelerated case, Eq. (7). Since one has the product of two oscillating functions, $\sin[\omega(t)\mu]$ and $J_1(V\mu)$, one can use a stationary phase approximation. The sine function oscillates with a phase $\phi_1 = \omega(t)\mu$, whereas the Bessel function $J_1$ oscillates with a phase $\phi_2 = V\mu$. Their product has thus two oscillating terms, one with a phase $\phi_3 = \phi_1 - \phi_2$. According to the stationary phase approximation, the important wave numbers are given by $d\phi_3/dk = 0$ and $d\phi_1/dk = 0$. The latter equation does not admit any real solution, and the corresponding contribution to the wave resistance decreases exponentially and can be neglected. The equation $d\phi_3/dk = 0$ leads to $[c_g(k) - V]\mu = 0$, where $c_g(k) = d\omega(k)/dk$ is the group velocity of the capillary-gravity waves. This equation has solutions only if $V \equiv \min[c_g(k) = (3\sqrt{3}/2 - 9/4)^{1/4}\min = 0.77 c_{\min}]$. When this condition is satisfied, two wave numbers are selected: $k_g$ (mainly dominated by gravity) and $k_c$ (mainly dominated by capillary forces), with $k_c < k_c$ (see Fig. 4). If these two wave numbers are sufficiently separated (that is, for velocities not too close to $0.77 c_{\min}$), one then finds that in the long time limit $R_u$ oscillates around its final value as

\[ R_u(t) = R_u(t) \cdot u_x \]

In the long time limit ($t \to \infty$), the Riemann–Lebesgue lemma, for a Lebesgue integrable function $f$

\[ \lim_{t \to \infty} \int f(x)e^{\varepsilon t}dx = 0 \]  

permits to determine the limit: the wave resistance given by Eq. (12) converges to 0 for $t \to \infty$, as expected. The behavior of $R_u(t) = R_u(t) \cdot u_x$ is represented in Fig. 3 for different
The wave resistance $R_w$ (in units of $P_r\Delta\gamma/\gamma$) is shown as a function of the reduced time $c_{min}\kappa t$ for an accelerated motion with different $U=V/c_{min}$ [see Eq. (7)]. Panels (a), (b), and (c) correspond to reduced velocities $U=1.5$, $1.1$, and $0.9$, respectively. The solid lines are obtained by a numerical integration of Eq. (7). The dashed lines correspond to the asymptotic expansion given by Eq. (14).

\[ R_w(t) = R_w(\infty) + \frac{1}{2\pi\rho\sqrt{V}} \frac{k_g^2/2}{\omega(k_g)} \left\{ \begin{array}{c} \cos(\Omega_c t) - 1 \end{array} \right\} \]

\[ + \frac{1}{2\pi\rho\sqrt{V}} \frac{k_g^2/2}{\omega(k_g)} \left\{ \begin{array}{c} \sin(\Omega_c t) - 1 \end{array} \right\}, \]

where $R_w(\infty)$ is given by Eq. (5) (and is equal to zero if $V < c_{min}$). $\Omega_c = [c_g(k_g) - c(k_g)]k_g$, and $\Omega_g = [c_g(k_g) - c(k_g)]k_g$.

Therefore, even if the disturbance velocity $V$ is smaller than $c_{min}$, there exists a transient nonzero wave resistance decreasing as $1/t$ (for $V > 0.77c_{min}$).

We obtain a good agreement between the numerical calculation of Eq. (7) and the analytical approximation, Eq. (14), as shown in Fig. 5. Note that the oscillations displayed by the wave resistance are characterized by a period $2\pi/\Omega_c$ or $2\pi/\Omega_g$ that diverges as $V$ approaches $c_{min}$. In the particular case where $V \gg c_{min}$, Eq. (14) reduces to

\[ R_w(t) = R_w(\infty) + \frac{1}{2\pi^2} \frac{k_g^2/2}{\rho V^2} \frac{\hat{P}_{ext}(k_g)^2}{\hat{P}_g(k_g)^2} \sin\left(\frac{c_{min}\kappa t}{8V}\right) \]

\[ - \frac{2\pi^2}{8V^3} \frac{\hat{P}_{ext}(k_g)^2}{\rho V t} \cos\left(\frac{8\pi^2}{27c_{min}}\right). \]

Let us now give a physical interpretation for the wave resistance behavior as described by Eq. (14): during the sudden acceleration of the disturbance (taking place at $t=0$), a large range of wavenumbers is emitted. Waves with wavenumbers $k$ such that $c_g(k) > V$ will move faster than the disturbance (which moves with the velocity $V$), whereas waves with wavenumbers $k$ such that $c_g(k) < V$ will move slower. The main interaction with the moving disturbance will therefore correspond to wavenumbers satisfying $c_g(k) = V$, that is, to $k$ and $k_g$ [hence their appearance in Eq. (14)]. Due to the Doppler effect, the wave resistance $R_w$ (which is the force exerted by the fluid on the moving disturbance) oscillates with an angular frequency $c_g(k) - V 27c_{min}$, hence the appearance of $\Omega_c$ and $\Omega_g$ in Eq. (14). Note that in the case of $V < 0.77c_{min}$, the wave resistance, Eq. (7), is nonzero and decreases exponentially with time [see Fig. 2(d)].

In order to get a better physical picture of the generated wave patterns, we have also calculated numerically the transient vertical displacement of the free surface $\zeta(r,t)$ in the accelerated case [Eq. (6)]. The corresponding patterns (as seen in the frame of the moving object) are presented in Figs. 6–8 for different reduced times $c_{min}\kappa t$ (1, 10, and 50, respectively) and a final velocity $V = 1.1c_{min}$. At $c_{min}\kappa t = 1$, the perturbation of the free surface is very localized around the disturbance (close to the same one obtained by a stone’s throw). At $c_{min}\kappa t = 10$, some capillary waves can already be observed at the front of the disturbance. At $c_{min}\kappa t = 50$, one observes a pattern that prefigur the steady pattern of the uniform straight motion (see Fig. 8 and Ref. 12, page 470). These predictions concerning the wave pattern might be compared to experimental data using the recent technique of Moisy et al.\textsuperscript{26}.

![Fig. 6. (Color online) Accelerated straight motion (final velocity $V = 1.1c_{min}$): transient vertical displacement [in units of $P_r\Delta\gamma/\rho g d^2$] of the free surface at $tc_{min}\kappa = 1$ obtained by inverse Fourier transform of Eq. (6). Note that the surface disturbance is localized around the object and close to the one obtained by a stone thrown in water.](image1)

![Fig. 7. (Color online) Accelerated straight motion (final velocity $V = 1.1c_{min}$): transient vertical displacement [in units of $P_r\Delta\gamma/\rho g d^2$] of the free surface at $tc_{min}\kappa = 10$ obtained by inverse Fourier transform of Eq. (6). One can already see ahead of the disturbance the waves associated with $k_1$, while the ones associated to $k_2$ are less well formed.](image2)
B. Decelerated straight motion

Let us now turn to the case of the decelerated motion described by Eq. (12). We first will assume that \( V > c_{\text{min}} \). In that case, the denominators appearing in the integrals vanish for \( k \) such that \( c(k) = V \), that is, for \( k_1 = \kappa (U^2 - \sqrt{U^4 - 1}) \) (mainly dominated by gravity) and \( k_2 = \kappa (U^2 + \sqrt{U^4 - 1}) \) (mainly dominated by capillary forces) as shown in Fig. 4, where \( U = V/c_{\text{min}} \). Both wavenumbers contribute to the wave resistance. Using the stationary phase approximation, one then finds that in the long time limit, \( R_w \) oscillates as \( \cos(Vk_1t + \pi/4)/\sqrt{t} \). More precisely, one has (for large \( t \))

\[
R_w(t) = \frac{1}{\sqrt{\pi V / \rho / \gamma}} \frac{k_1^{3/2} \tilde{P}_{\text{ext}}(k_1)}{(k_2 - k_1) c_g} \cos(Vk_1t + \pi/4). \tag{16}
\]

The wave resistance displays oscillations that are characterized by a period of \( 2\pi/(V k_1) \). Figure 9 shows the good agreement between the numerical calculation of Eq. (12) and the analytical approximation, Eq. (16), for large times (\( c_{\text{min}} k_1 t > 10 \)). Again one can give a simple physical interpretation of the wave resistance, Eq. (16). The period of the oscillations is given by \( 2\pi/(V k_1) \) and depends only on the wavenumber \( k_1 \) (mainly dominated by gravity) even in the case of an object with a size \( b \) much smaller than the capillary length. This can be understood as follows. For \( t < 0 \), the disturbance moves with a constant velocity \( V \) and emits waves with wavenumber \( k_1 \) and waves with wavenumber \( k_2 \) (mainly dominated by capillary forces). The \( k_1 \)-wavelags behind the disturbance, while the \( k_2 \)-waves move ahead of it. When the disturbance stops at \( t = 0 \), the \( k_2 \)-waves keep moving forward and do not interact with the disturbance. However, the \( k_1 \)-waves will encounter the disturbance and interact with it, hence the period \( 2\pi/(V k_1) \) of the wave resistance, Eq. (16). The \( 1/\sqrt{t} \) decrease in the magnitude of wave resistance in Eq. (16) can be understood as follows. At time \( t > 0 \), the disturbance (which is at rest at \( x = 0 \)) is hit by a \( k_1 \)-wave that has previously been emitted by a distance of \( c_g k_1 t \) away from it. The vertical amplitude \( \zeta \) of this wave is inversely proportional to the square root of this distance. Indeed, as the liquid is inviscid, the energy has to be conserved, and one thus has \( \zeta \propto 1/\sqrt{c_g k_1 t} \). Since the wave resistance \( R_w \) is proportional to vertical amplitude of the wave \( \zeta \), one recovers that \( R_w \) decreases with time as \( 1/\sqrt{c_g(k_1)t} \).

In the particular case where \( V > c_{\text{min}} \), Eq. (16) reduces to

\[
R_w(t) = \frac{1}{2\sqrt{\pi}} \kappa^{3/2} \tilde{P}_{\text{ext}}(k_1) c_g^{3/2} \cos \left( \frac{k_1 c_{\text{min}} t}{V} + \frac{\pi}{4} \right) \tag{17}
\]

Let us now discuss the case of \( V < c_{\text{min}} \) (still considering the decelerated motion, Eq. (12)). In that case, the wave resistance oscillates (in the long time limit) approximately as

\[
e^{-U^4/4} \left( \sqrt{1 - U^2} \right) \sum_{k} c_{\text{min}} \kappa \sin(U^4 \kappa c_{\text{min}} + \chi).
\]

More precisely, one has (for large \( t \))

\[
R_w(t) = \sqrt{\frac{2}{\pi}} \kappa^{3/2} \tilde{P}_{\text{ext}}(k_1) / \rho c_{\text{min}}^{3/2} \sqrt{c_{\text{min}} c_{\text{t}}}
\times e^{-U^4/4} \left( 1 - U^2 \right)^{1/4} (3U^4 + 1)^{1/4}, \tag{18}
\]

where

\[
\chi = (3/2) \arctan \left( \sqrt{1 - U^4/U^2} \right)
- (1/2) \arctan \left( \sqrt{1 - U^4/(2U^2)} \right).
\tag{19}
\]

and \( \tilde{k} = \kappa (U^2 + \sqrt{U^4 - 1}) \). Note that in the case of \( V < c_{\text{min}} \), the wave resistance, Eq. (18), also displays some oscillations, although no waves were emitted at \( t < 0 \). These oscillations that have an exponential decay in time might be due to the sudden arrest of the disturbance. Such oscillations could also be present in the case of \( V > c_{\text{min}} \) but are hidden by the interaction between the \( k_1 \)-waves and the disturbance, which lead to a much more slower \( 1/\sqrt{t} \) decay.

IV. CONCLUSIONS

In this article, we have shown that a disturbance undergoing a rectilinear accelerated or decelerated motion at a liquid-air interface emits waves even if its velocity \( V \) (the final one in the accelerated case and the initial one in the decelerated case, respectively) is smaller than \( c_{\text{min}} \). This cor-
borobates the results in Ref. 13. For this purpose, we treat the wave emission problem by a linearized theory in a Monge representation. Then, we derive the analytical expression of the wave resistance and solve it by numerical integration. Asymptotic expansions permit to extract the predominant behavior of the wave resistance. Some vertical displacement patterns are also calculated in order to shown how the waves invade the free surface.

The results presented in this paper should be important for a better understanding of the propulsion of water-walking insects,27–30 such as whirligig beetles, where accelerated and decelerated motions frequently occur (e.g., when hunting a prey or escaping a predator).31 Even in the case where the insect motion appears as rectilinear and uniform, one has to keep in mind that the rapid leg strokes are accelerated and might produce a wave drag even below $c_{\text{min}}$. The predictions concerning the wave patterns might be compared to experimental data using the recent technique of Moisy et al.26

It will be interesting to take in our model some nonlinear effects because the waves radiated by whirligig beetles have a large amplitude. Recently Chepelianski et al. derived a self-consistent integral equation describing the flow velocity around the moving disturbance.33 It would be interesting to incorporate this approach into the present study.

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19The Fourier transform $\tilde{f}(k,\omega)$ is related to the function $f(\mathbf{r},t)$ through

$$f(\mathbf{r},t)=\int \frac{d^dk}{(2\pi)^d} \mathcal{P} \tilde{u}(k) \delta(\mathbf{r} - \mathbf{r}_k).$$

25In the particular case of a Lorentzian pressure disturbance with a Fourier transform $P_{\omega}(k) = \rho_0 e^{-k^2}$, where $b$ is the object size (set to $b = 0.1 \kappa^2$), we have numerically checked that up to $V \sim 8 \kappa_{\text{min}}$ the main contribution to the wave resistance is due to $k_1$.
34See, e.g., Ref. 12. The crest pattern obtained by this argument is characterized up to arbitrary translations along the $x$-axis.

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