Negative-weight percolation

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Outline

- Introduction
- Percolation problem
- Results
- Summary

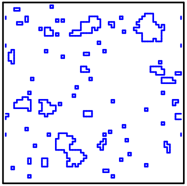
Model

- $L \times L$ lattice, fully periodic boundary conditions
- Undirected edges, weight (cost) distribution:

$$P(\omega) = \rho\,(2\pi)^{-1/2}\exp(-\omega^2/2) \;+\; (1-\rho)\,\delta(\omega-1)$$

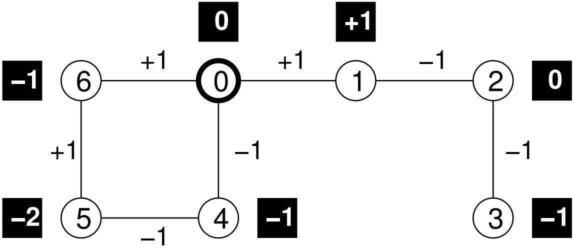
- Allows for loops $\mathcal{L}$ with negative weight $\omega_{\mathcal{L}}$
- Agent on lattice edges: pay/receive resources
- Configuration $\mathcal{C}$ of loops, with

$$E \equiv \sum_{\mathcal{L}\in\mathcal{C}} \omega_{\mathcal{L}} \stackrel{!}{=} \min$$



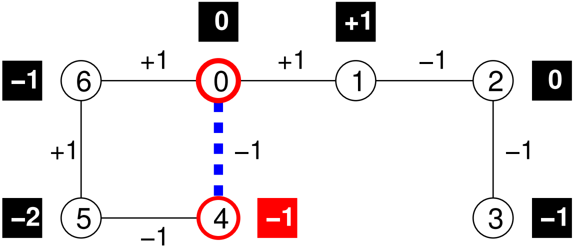
- Obtain $\mathcal{C}$ through mapping to minimum weight perfect matching problem [O. Melchert & AKH, New J. Phys. 2008]

Minimal distances



- $d(i) = \min_{j \in N(i)}(d(j) + \omega(i,j))$ not fulfilled

Minimal distances



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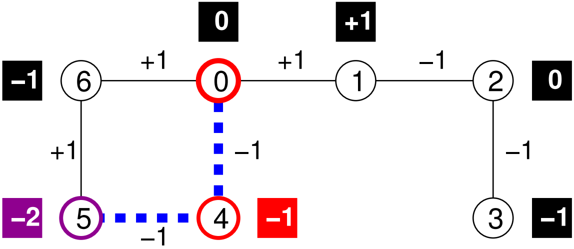

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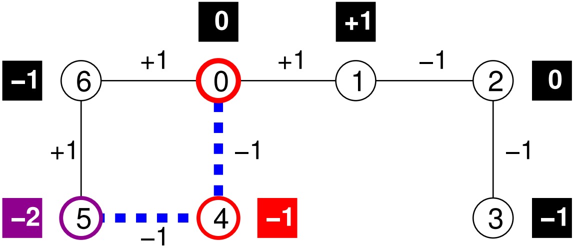

- $d(i) = \min_{j \in N(i)}(d(j) + \omega(i,j))$ not fulfilled
- Standard minimum-weight path algorithms, e.g. Dijkstra, Bellman-Ford, Floyd-Warshall, don't work

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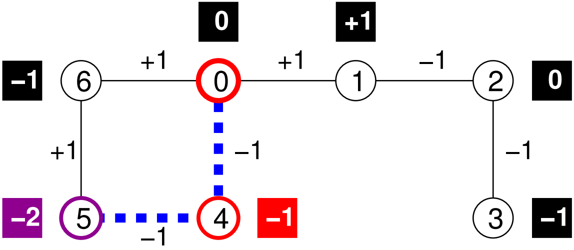

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- Standard minimum-weight path algorithms, e.g. Dijkstra, Bellman-Ford, Floyd-Warshall, don't work
- Minimum-weight path problem requires matching techniques

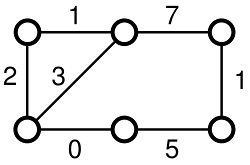
  [R.K. Ahuja, T.L. Magnanti and J.B. Orlin, *Network flows*]

Algorithm – Outline

Brief description of the basic steps:

- Set up auxiliary graph
- Find minimum weight perfect matching (MWPM) on auxiliary graph
- Interpret MWPM as minimal weighted set of paths/loops

- Graph $G = (V, E)$:



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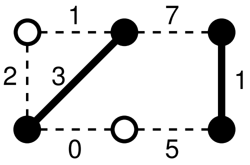
- Matching $M \subset E$:

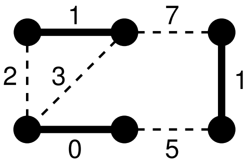


Algorithm – Outline

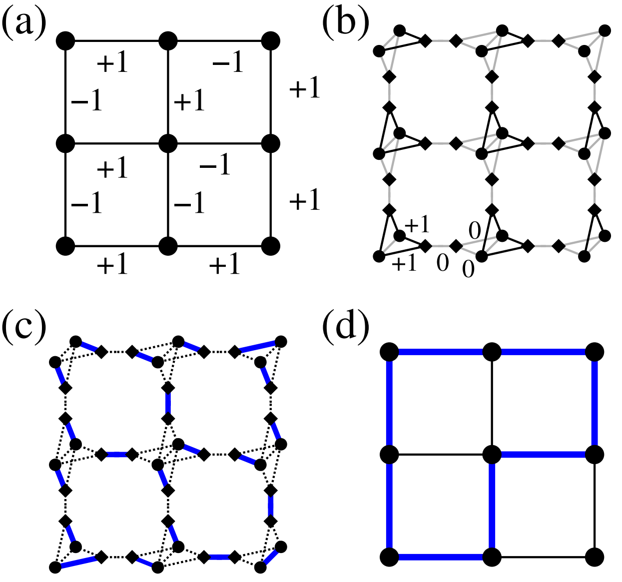
Brief description of the basic steps:

- Set up auxiliary graph
- Find minimum weight perfect matching (MWPM) on auxiliary graph
- Interpret MWPM as minimal weighted set of paths/loops

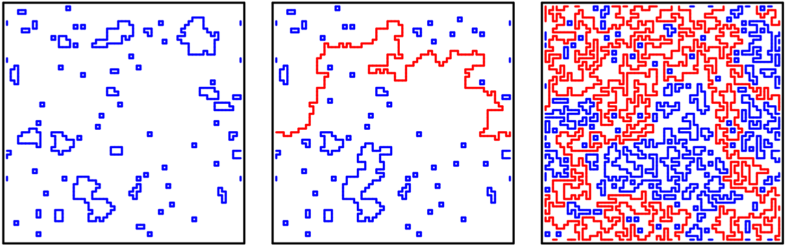
- MWPM, $\omega_M = 2$



Algorithm – Mapping procedure



Loop percolation

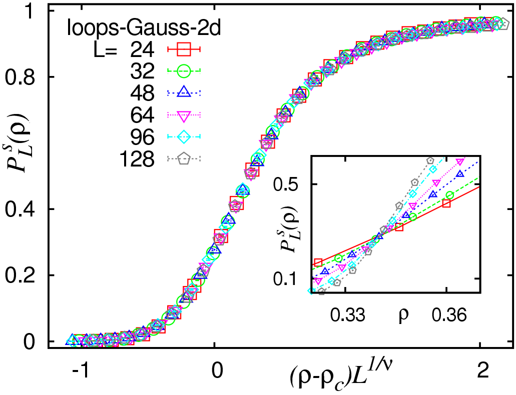


($L = 64$ at $\rho = 0.335,\ 0.340,\ 0.750$)

- Observe system spanning loops above critical ρ
- Disorder induced, geometric transition
- Characterize loops using observables from percolation theory (finite-size scaling (FSS) analysis)

  [D. Stauffer, A. Aharony, *Introduction to Percolation Theory*]

Percolation probability



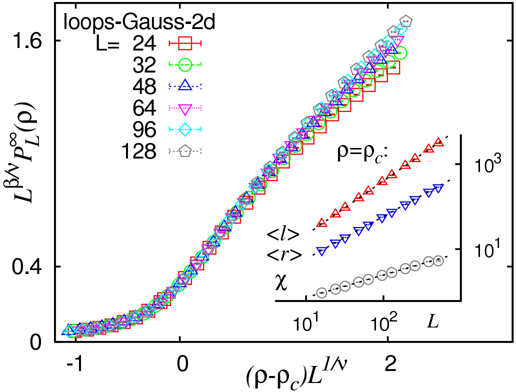
Percolation probability exhibits FSS:

$P_L^s \sim f[(\rho-\rho_c)L^{1/\nu}]$

$\rho_c = 0.340(1)$
$\nu = 1.49(7)$
(rand. perc.: $\nu = 1.33$)
$S = 0.91$

- S = "quality" of the scaling assumption
- Similar scaling for mean number of spanning loops

Percolation strength



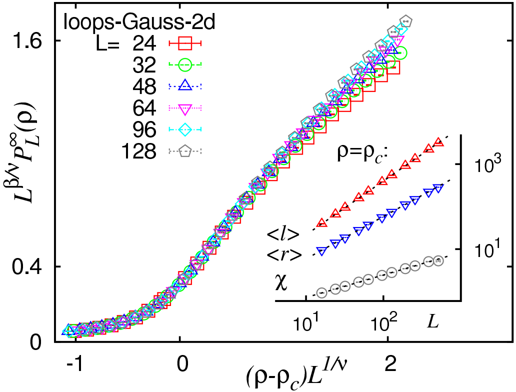
Exhibits FSS:

$P_L^\infty \sim L^{-\beta/\nu} f[(\rho-\rho_c)L^{1/\nu}]$

$\beta = 1.07(6)$
$S = 1.16$

- Probability $P_L^\infty \equiv \langle \ell \rangle / L^d$ that edge belongs to percolating loop, finite-size suszeptibility $\chi \equiv L^{-d}(\langle \ell^2 \rangle - \langle \ell \rangle^2)$

Percolation strength



At ρ_c ($L_{max} = 512$):

loop length $\langle \ell \rangle \sim L^{d_f}$,
roughness $\langle r \rangle \sim L^{d_r}$,
suszept. $\chi \sim L^{\gamma/\nu}$

$d_f = 1.266(2)$
$d_r = 1.001(4)$
$\gamma = 0.77(7)$

- Probability $P_L^{\infty} \equiv \langle \ell \rangle / L^d$ that edge belongs to percolating loop, finite-size suszeptibility $\chi \equiv L^{-d}(\langle \ell^2 \rangle - \langle \ell \rangle^2)$
- Scaling relations $d_f = d - \beta/\nu$ and $\gamma + 2\beta = d\nu$ are fulfilled

Small loops – Scaling at ρ_c

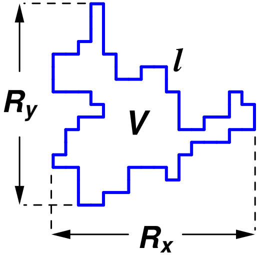
Results for non-spanning loops at ρ_c for $L = 256$

- coarse measure for loop shape:

  spanning length: $R \equiv \max(R_x, R_y)$

$$\frac{R_x R_y}{2(R_x + R_y)} = aR^b$$

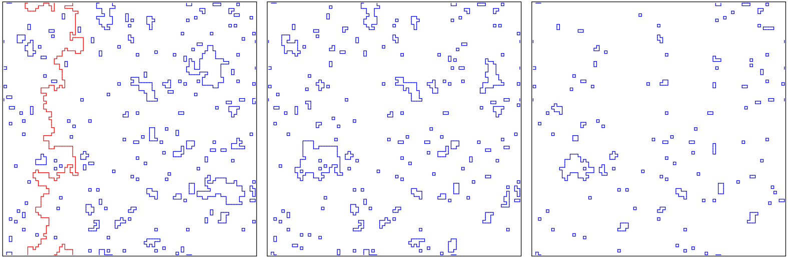
loops:	$a = 0.216(1)$	$b = 1.000(1)$
square:	$a = 0.25$	$b = 1.0$



- scaling properties:

fractal surface	$\langle R \rangle \sim \ell^{1/\tilde{d}_f}$	$\tilde{d}_f = 1.267(1)$
loop roughness	$\langle r \rangle \sim R^{\tilde{d}_r}$	$\tilde{d}_r = 0.999(2)$
cluster volume	$\langle v \rangle \sim \ell^{\tilde{d}_v/\tilde{d}_f}$	$\tilde{d}_v = 1.99(1)$
loop weight	$\langle \omega \rangle \sim \ell$	

Small loops – Length distribution



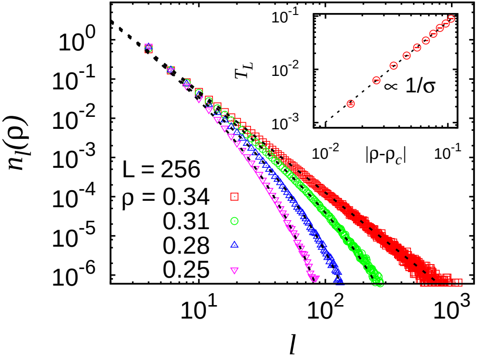
$(L = 96,\ \rho = 0.34, 0.324, 0.28)$

- $\rho < \rho_c$: loops with large perimeter are suppressed
- Investigate distribution of loop lengths close to ρ_c

[D. Stauffer, *Scaling Theory of Percolation Clusters*, Phys. Rep. (1979)]

Small loops – Length distribution

- Distribution $n_\ell(\rho)$ of the loop lengths ℓ for $L = 256$



Expected FSS:

$n_\ell(\rho) \sim \ell^{-\tau} \exp(-T_L \ell)$

Loop tension:

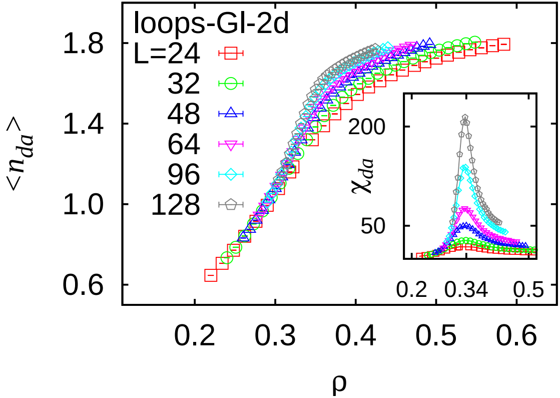
$T_L \sim |\rho - \rho_c|^{1/\sigma}$

$\tau = 2.59(3)$
$\sigma = 1.89(7)$

- At ρ_c: cut-off scale $\ell_0 = 1/T_L$ for loop lengths diverges
- Consistent with scaling relations $\tau = 1 + d/d_f$ and $\nu d_f = 1/\sigma$

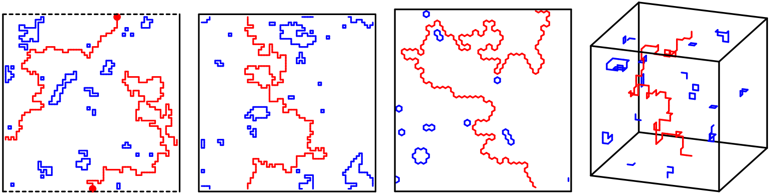
Running time

- Measure for running time: number n_{da} of solution adjustments per lattice site



- Corresponding fluctuations are peaked close to ρ_c

More results



Type	ρ_c	ν	β	γ	τ	d_f
P±J 2d sq	0.1032(5)	1.43(6)	1.03(3)	0.76(5)	2.51(4)	1.268(1)
L±J 2d sq	0.1028(3)	1.49(9)	1.09(8)	0.75(8)	2.58(6)	1.260(2)
L±J 2d hex	0.1583(6)	1.47(9)	1.07(9)	0.76(8)	2.59(2)	1.264(3)
L-Gl 2d sq	0.340(1)	1.49(7)	1.07(6)	0.77(7)	2.59(3)	1.266(2)
L±J 3d cu	0.0286(1)	1.02(3)	1.80(8)	–	3.5(3)	1.30(1)

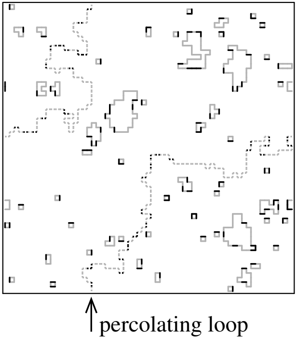
[OM & A.K. Hartmann, NJP 10, 043039 (2008)]

- Random bond Ising model at $T = 0$:
  $\rho_c = 0.103(1)$, $\nu = 1.55(1)$, $\beta = 0.09(1)$ (±J) [Amoruso & AKH, PRB 2004]
  $\rho_c = 0.340(1)$, $\nu = 1.49(7)$, $\beta = 0.097(6)$ (Gl) [LA, OM & AKH, in prep.]

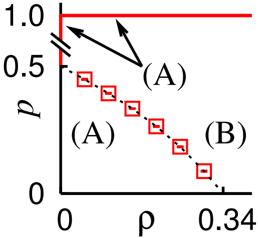
Effect of dilution

- Study effect of dilution on critical properties in 2*d*
- Consider two types of dilution:

Type I: fraction *p* of edges with $\omega = 0$



Phase diagram:



- Critical exponents remain the same

[L. Apolo, O. Melchert & AKH, PRE 2009]

Effect of dilution

- Study effect of dilution on critical properties
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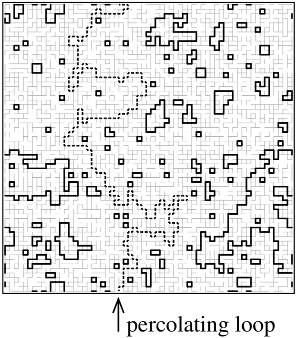
Type II: fraction *p* of absent edges (bond percolation)



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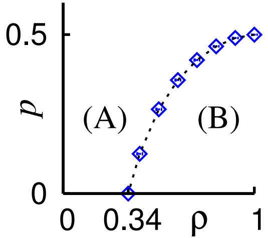

- Critical exponents change

[L. Apolo, O. Melchert & AKH, PRE 2009]

Summary

- Negative-weight percolation of loops
- Distinct from random bond/site percolation
- $2d$: critical exponents close to RBIM
- More details: OM & A.K. Hartmann, Phys. Rev. E 79, 031103 (2009)

Summary

- Negative-weight percolation of loops
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- 2*d*: critical exponents close to RBIM
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- Thank you!